

# Classical Integrable Super sinh-Gordon equation with Defects

J.F. Gomes, L.H. Ymai and A.H. Zimerman

Instituto de Física Teórica-UNESP  
Rua Pamplona 145  
01405-900 São Paulo, Brazil

## Abstract

The introduction of defects is discussed under the Lagrangian formalism and Backlund transformations for the  $N = 1$  super sinh-Gordon model. Modified conserved momentum and energy are constructed for this case. Some explicit examples of different Backlund solitons solutions are discussed. The Lax formulation within the space split by the defect leads to the integrability of the model and henceforth to the existence of an infinite number of constants of motion.

## 1 Introduction

A quantum integrable theory of defects involving free bosonic and free fermionic fields was first studied in ref. [1] following the achievements obtained in studying the quantum field theory with boundaries [2], [3]. The Lagrangian formulation of a class of relativistic integrable field theories admitting certain discontinuities has been studied recently [4] - [6]. In particular, in ref. [4] the authors have considered a field theory in which different soliton solutions of the sine-Gordon model are linked in such a way that the integrability is preserved. The integrability of the total system imposes severe constraints specifying the possible types of defects. These are characterized by Backlund transformations which are known to connect two different soliton solutions.

The supersymmetric  $N = 1$  sinh-Gordon in terms of superfields was proposed in ref. [7] by introducing a pair of Grassmann coordinates. The corresponding Backlund transformation was proposed in terms of superfields also. An important piece of information characterizing the defect is given by boundary functions which are consistent with the Backlund transformations and leads to the construction of modified conserved momentum and energy. The aim of this paper is to consider a classical interacting field theory containing both Bose and Fermi fields. We generalize the Lagrangian approach of [4] - [6] to include the  $N = 1$  supersymmetric sinh-Gordon in the presence of integrable defects described by Backlund transformation. We explicitly construct the boundary functions consistently with the Backlund transformations. Several cases of transition through the defect are studied. In particular, when the fermionic fields vanish, the pure bosonic soliton case of refs. [4] - [6] is recovered. On the other hand, for vanishing bosonic fields, we find the pure fermionic case. More interesting cases occur when we consider transitions between vacuum to a system of Bose and Fermi fields and transitions between two distinct configurations of non trivial

Bose and Fermi states. The integrability of the system is ensured by the zero curvature representation of the equations of motion.

This paper is organized as follows. In Sect. 2 we discuss the Lagrangian approach to describe these integrable supersymmetric models with defects. In particular, specific boundary terms are chosen in order to ensure modified energy momentum conservation. Explicit examples of pure free bosonic and free fermionic fields together with the supersymmetric sinh-Gordon are also discussed. Various Backlund solutions for the super sinh-Gordon (sine-Gordon) models are discussed in Sect.3. In Sect. 4 we present the zero curvature representation of the supersymmetric sinh-Gordon model equations of motion. By introducing two regions around the defect [4] we explicitly construct, in a closed form, a gauge group element connecting the Lax pair in the overlap region. This fact guarantees the existence of an infinite set of conservation laws. In the appendix we present, in components the Backlund transformation for the super sinh-Gordon model.

## 2 Integrable Supersymmetric $sl(2, 1)$ Field Theory with a Defect

### 2.1 Lagrangian Description

The starting point is the Lagrangian density describing bosonic,  $\phi_1$  and fermionic,  $\bar{\psi}_1, \psi_1$  fields in the region  $x < 0$  and correspondingly  $\phi_2$  and  $\bar{\psi}_2, \psi_2$  in the region  $x > 0$ . A defect contribution located at  $x = 0$  can be introduced such that the model under study is described by the Lagrangian density

$$\mathcal{L} = \theta(-x)\mathcal{L}_1 + \theta(x)\mathcal{L}_2 + \delta(x)\mathcal{L}_D, \quad (2.1)$$

where

$$\begin{aligned} \mathcal{L}_p &= \frac{1}{2}(\partial_x \phi_p)^2 - \frac{1}{2}(\partial_t \phi_p)^2 - \bar{\psi}_p \partial_x \bar{\psi}_p + \bar{\psi}_p \partial_t \bar{\psi}_p + \psi_p \partial_x \psi_p + \psi_p \partial_t \psi_p \\ &\quad + V_p(\phi_p) + W_p(\phi_p, \psi_p, \bar{\psi}_p), \quad p = 1, 2 \\ \mathcal{L}_D &= \frac{1}{2}(\phi_2 \partial_t \phi_1 - \phi_1 \partial_t \phi_2) - \psi_1 \psi_2 - \bar{\psi}_1 \bar{\psi}_2 + 2f_1 \partial_t f_1 \\ &\quad + B_0(\phi_1, \phi_2) + B_1(\phi_1, \phi_2, \psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2, f_1), \end{aligned} \quad (2.2)$$

$f_1$  is an auxiliary fermionic field and  $V_p, W_p$  correspond to field potentials. The quantities  $B_0$  and  $B_1$  are boundary functions describing the defect. The field equations in the two regions together with the defect conditions at  $x = 0$  are then given by

$$\begin{aligned} \partial_x^2 \phi_1 - \partial_t^2 \phi_1 &= \partial_{\phi_1} V_1 + \partial_{\phi_1} W_1, \\ \partial_x \psi_1 + \partial_t \bar{\psi}_1 &= -\frac{1}{2} \partial_{\psi_1} W_1, \\ -\partial_x \bar{\psi}_1 + \partial_t \psi_1 &= -\frac{1}{2} \partial_{\bar{\psi}_1} W_1, \quad x < 0 \end{aligned} \quad (2.3)$$

and

$$\begin{aligned}
\partial_x^2 \phi_2 - \partial_t^2 \phi_2 &= \partial_{\phi_2} V_2 + \partial_{\phi_2} W_2, \\
\partial_x \psi_2 + \partial_t \psi_2 &= -\frac{1}{2} \partial_{\psi_2} W_2, \\
-\partial_x \bar{\psi}_2 + \partial_t \bar{\psi}_2 &= -\frac{1}{2} \partial_{\bar{\psi}_2} W_2, \quad x > 0
\end{aligned} \tag{2.4}$$

For  $x = 0$  we find

$$\partial_x \phi_1 - \partial_t \phi_2 = -\partial_{\phi_1} B_0 - \partial_{\phi_1} B_1, \tag{2.5}$$

$$\partial_x \phi_2 - \partial_t \phi_1 = \partial_{\phi_2} B_0 + \partial_{\phi_2} B_1, \tag{2.6}$$

$$\psi_1 + \psi_2 = \partial_{\psi_1} B_1 = -\partial_{\psi_2} B_1, \tag{2.7}$$

$$\bar{\psi}_1 - \bar{\psi}_2 = -\partial_{\bar{\psi}_1} B_1 = -\partial_{\bar{\psi}_2} B_1, \tag{2.8}$$

$$\partial_t f_1 = -\frac{1}{4} \partial_{f_1} B_1. \tag{2.9}$$

with  $\partial_{\psi_p} = \frac{\partial}{\partial \psi_p}$  are fermionic derivatives acting on the left (the same holds for  $\partial_{\bar{\psi}_p}$  and  $\partial_{f_1}$ ).

Let us first consider the time derivative of the momentum:

$$\begin{aligned}
\frac{dP}{dt} &= \frac{d}{dt} \left[ \int_{-\infty}^0 dx \left( \partial_t \phi_1 \partial_x \phi_1 - \bar{\psi}_1 \partial_x \bar{\psi}_1 - \psi_1 \partial_x \psi_1 \right) \right. \\
&\quad \left. + \int_0^{+\infty} dx \left( \partial_t \phi_2 \partial_x \phi_2 - \bar{\psi}_2 \partial_x \bar{\psi}_2 - \psi_2 \partial_x \psi_2 \right) \right].
\end{aligned} \tag{2.10}$$

From the equations (2.3), (2.4) and ignoring contributions from  $\pm\infty$ , we can write:

$$\begin{aligned}
\frac{dP}{dt} &= \left[ \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_t \phi_1)^2 - \bar{\psi}_1 \partial_t \bar{\psi}_1 - \psi_1 \partial_t \psi_1 - V_1 - W_1 \right. \\
&\quad \left. - \frac{1}{2} (\partial_x \phi_2)^2 - \frac{1}{2} (\partial_t \phi_2)^2 + \bar{\psi}_2 \partial_t \bar{\psi}_2 + \psi_2 \partial_t \psi_2 + V_2 + W_2 \right]_{x=0}.
\end{aligned} \tag{2.11}$$

Using the boundary conditions (2.5)-(2.8) and assuming that

$$\frac{1}{2} (\partial_{\phi_1} B_0)^2 - \frac{1}{2} (\partial_{\phi_2} B_0)^2 - V_1 + V_2 = 0, \tag{2.12}$$

$$(\partial_{\phi_1} B_1)^2 = (\partial_{\phi_2} B_1)^2 = 0, \tag{2.13}$$

equation (2.11) takes the following form

$$\begin{aligned}
\frac{dP}{dt} &= \left[ -(\partial_t \phi_2 \partial_{\phi_1} + \partial_t \phi_1 \partial_{\phi_2})(B_0 + B_1) + \partial_{\phi_1} B_0 \partial_{\phi_1} B_1 - \partial_{\phi_2} B_0 \partial_{\phi_2} B_1 \right. \\
&\quad - \partial_t \bar{\psi}_1 \partial_{\bar{\psi}_1} B_1 - \partial_t \bar{\psi}_2 \partial_{\bar{\psi}_2} B_1 + \partial_t \psi_1 \partial_{\psi_1} B_1 + \partial_t \psi_2 \partial_{\psi_2} B_1 \\
&\quad \left. - W_1 + W_2 + \partial_t (\bar{\psi}_1 \bar{\psi}_2) - \partial_t (\psi_1 \psi_2) \right]_{x=0}.
\end{aligned} \tag{2.14}$$

Introducing new variables

$$\phi_{\pm} = \phi_1 \pm \phi_2 \quad \rightarrow \quad \begin{cases} \partial_{\phi_1} = \partial_{\phi_+} + \partial_{\phi_-} \\ \partial_{\phi_2} = \partial_{\phi_+} - \partial_{\phi_-} \end{cases} \quad (2.15)$$

$$\bar{\psi}_{\pm} = \bar{\psi}_1 \pm \bar{\psi}_2 \quad \rightarrow \quad \begin{cases} \partial_{\bar{\psi}_1} = \partial_{\bar{\psi}_+} + \partial_{\bar{\psi}_-} \\ \partial_{\bar{\psi}_2} = \partial_{\bar{\psi}_+} - \partial_{\bar{\psi}_-} \end{cases} \quad (2.16)$$

$$\psi_{\pm} = \psi_1 \pm \psi_2 \quad \rightarrow \quad \begin{cases} \partial_{\psi_1} = \partial_{\psi_+} + \partial_{\psi_-} \\ \partial_{\psi_2} = \partial_{\psi_+} - \partial_{\psi_-} \end{cases} \quad (2.17)$$

we can see from the equations (2.7) and (2.8) that

$$\partial_{\psi_+} B_1 = 0, \quad \partial_{\bar{\psi}_-} B_1 = 0. \quad (2.18)$$

The above conditions suggest that  $B_1$  is independent of  $\psi_+$  and  $\bar{\psi}_-$ .

Let us assume that

$$\partial_{\phi_+} \partial_{\phi_-} B_0 = 0, \quad \partial_{\phi_+} \partial_{\phi_-} B_1 = 0, \quad \partial_{\bar{\psi}_+} \partial_{\psi_-} B_1 = 0, \quad (2.19)$$

so that we decompose

$$B_0 = B_0^+(\phi_+) + B_0^-(\phi_-), \quad (2.20)$$

$$B_1 = B_1^+(\phi_+, \bar{\psi}_+, f_1) + B_1^-(\phi_-, \psi_-, f_1). \quad (2.21)$$

In terms of the new variables and using (2.20), (2.21) and (2.9), equation (2.14) is written as

$$\begin{aligned} \frac{dP}{dt} &= \left[ \partial_t (-B_0^+ + B_0^- - B_1^+ + B_1^- + \bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2) \right]_{x=0} \\ &+ \left[ \frac{1}{2} \partial_{f_1} B_1^+ \partial_{f_1} B_1^- + 2 \partial_{\phi_+} B_0^+ \partial_{\phi_-} B_1^- + 2 \partial_{\phi_-} B_0^- \partial_{\phi_+} B_1^+ - W_1 + W_2 \right]_{x=0}, \end{aligned} \quad (2.22)$$

which reduces to a total time derivative, provided

$$\frac{1}{2} \partial_{f_1} B_1^+ \partial_{f_1} B_1^- + 2 \partial_{\phi_+} B_0^+ \partial_{\phi_-} B_1^- + 2 \partial_{\phi_-} B_0^- \partial_{\phi_+} B_1^+ = W_1 - W_2. \quad (2.23)$$

Thus, the combination

$$\mathcal{P} = P + \left[ (B_0^+ - B_0^-) + (B_1^+ - B_1^-) - \bar{\psi}_1 \bar{\psi}_2 + \psi_1 \psi_2 \right]_{x=0} \quad (2.24)$$

is conserved. For the energy:

$$\begin{aligned} E &= \int_{-\infty}^0 dx \left[ \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_t \phi_1)^2 - \bar{\psi}_1 \partial_x \bar{\psi}_1 + \psi_1 \partial_x \psi_1 + V_1 + W_1 \right] \\ &+ \int_0^{\infty} dx \left[ \frac{1}{2} (\partial_x \phi_2)^2 + \frac{1}{2} (\partial_t \phi_2)^2 - \bar{\psi}_2 \partial_x \bar{\psi}_2 + \psi_2 \partial_x \psi_2 + V_2 + W_2 \right], \end{aligned} \quad (2.25)$$

the conserved quantity is given by the combination

$$\mathcal{E} = E + \left[ (B_0^+ + B_0^-) + (B_1^+ + B_1^-) - \bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2 \right]_{x=0}. \quad (2.26)$$

## 2.2 Bosonic free field theory

Consider the case where the fermionic fields vanish and let us recall the results of ref. [5]. The Lagrangian density in the regions  $x < 0$  and  $x > 0$  are given by

$$\mathcal{L}_p = \frac{1}{2}(\partial_x \phi_p)^2 - \frac{1}{2}(\partial_t \phi_p)^2 + V_p, \quad p = 1, 2 \quad (2.27)$$

where

$$V_p = \frac{1}{2}m^2\phi_p^2, \quad p = 1, 2. \quad (2.28)$$

The field equations are

$$\partial_x^2 \phi_p - \partial_t^2 \phi_p = m^2 \phi_p \quad p = 1, 2. \quad (2.29)$$

At  $x = 0$  the Lagrangian density associated with the defect is

$$\mathcal{L}_D = \frac{1}{2}(\phi_2 \partial_t \phi_1 - \phi_1 \partial_t \phi_2) + B_0(\phi_1, \phi_2). \quad (2.30)$$

A solution satisfying the equation (2.12) and the first equation of (2.19) is given by

$$B_0 = -\frac{m\beta^2}{4}(\phi_1 - \phi_2)^2 - \frac{m}{4\beta^2}(\phi_1 + \phi_2)^2, \quad (2.31)$$

where  $\beta$  is a free parameter. Thus, the defect conditions at  $x = 0$ , namely (2.5) and (2.6) become

$$\begin{aligned} \partial_x \phi_1 - \partial_t \phi_2 &= \frac{m\beta^2}{2}(\phi_1 - \phi_2) + \frac{m}{2\beta^2}(\phi_1 + \phi_2), \\ \partial_x \phi_2 - \partial_t \phi_1 &= \frac{m\beta^2}{2}(\phi_1 - \phi_2) - \frac{m}{2\beta^2}(\phi_1 + \phi_2). \end{aligned} \quad (2.32)$$

The modified conserved momentum and energy are, respectively

$$\begin{aligned} \mathcal{P} &= P + \left[ -\frac{m\beta^2}{4}(\phi_1 - \phi_2)^2 + \frac{m}{4\beta^2}(\phi_1 + \phi_2)^2 \right]_{x=0}, \\ \mathcal{E} &= E + \left[ -\frac{m\beta^2}{4}(\phi_1 - \phi_2)^2 - \frac{m}{4\beta^2}(\phi_1 + \phi_2)^2 \right]_{x=0} \end{aligned} \quad (2.33)$$

## 2.3 Fermionic free field theory

Let us consider the following Lagrangian

$$\mathcal{L}_p = -\bar{\psi}_p \partial_x \bar{\psi}_p + \bar{\psi}_p \partial_t \bar{\psi}_p + \psi_p \partial_x \psi_p + \psi_p \partial_t \psi_p + W_p(\psi_p, \bar{\psi}_p), \quad (2.34)$$

where,

$$W_p = 2m\bar{\psi}_p \psi_p, \quad p = 1, 2. \quad (2.35)$$

Then, the field equations in the regions  $x < 0$  and  $x > 0$  are

$$\partial_x \psi_p + \partial_t \psi_p = m \bar{\psi}_p, \quad \partial_x \bar{\psi}_p - \partial_t \bar{\psi}_p = m \psi_p, \quad p = 1, 2 \quad (2.36)$$

The Lagrangian associated with the defect is taken to be

$$\mathcal{L}_D = -\psi_1 \psi_2 - \bar{\psi}_1 \bar{\psi}_2 + 2f_1 \partial_t f_1 + B_1(\psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2, f_1). \quad (2.37)$$

where

$$B_1 = -\frac{2i}{\beta} \sqrt{m} f_1 (\bar{\psi}_1 + \bar{\psi}_2) + i\beta \sqrt{m} f_1 (\psi_1 - \psi_2), \quad (2.38)$$

satisfies the condition required by the conservation of the modified momentum and energy (2.19) and (2.23), i.e.,

$$\partial_{\bar{\psi}_+} \partial_{\psi_-} B_1 = 0, \quad \frac{1}{2} \partial_{f_1} B_1^+ \partial_{f_1} B_1^- = W_1 - W_2. \quad (2.39)$$

The defect conditions at  $x = 0$  are then

$$\begin{aligned} \psi_1 + \psi_2 &= -i\beta \sqrt{m} f_1, & \bar{\psi}_1 - \bar{\psi}_2 &= -\frac{2i}{\beta} \sqrt{m} f_1, \\ \partial_t f_1 &= \frac{i}{2\beta} \sqrt{m} (\bar{\psi}_1 + \bar{\psi}_2) - \frac{i\beta}{4} \sqrt{m} (\psi_1 - \psi_2). \end{aligned} \quad (2.40)$$

and the modified conserved momentum and energy are given by

$$\begin{aligned} \mathcal{P} &= P + \left[ -\frac{2i}{\beta} \sqrt{m} f_1 (\bar{\psi}_1 + \bar{\psi}_2) - i\beta \sqrt{m} f_1 (\psi_1 - \psi_2) - \bar{\psi}_1 \bar{\psi}_2 + \psi_1 \psi_2 \right]_{x=0} \\ \mathcal{E} &= E + \left[ -\frac{2i}{\beta} \sqrt{m} f_1 (\bar{\psi}_1 + \bar{\psi}_2) + i\beta \sqrt{m} f_1 (\psi_1 - \psi_2) - \bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2 \right]_{x=0}. \end{aligned} \quad (2.41)$$

## 2.4 Supersymmetric Sinh-Gordon

Consider the Lagrangian density (2.1) with

$$\begin{aligned} \mathcal{L}_p &= \frac{1}{2} (\partial_x \phi_p)^2 - \frac{1}{2} (\partial_t \phi_p)^2 - \bar{\psi}_p \partial_x \bar{\psi}_p + \bar{\psi}_p \partial_t \bar{\psi}_p + \psi_p \partial_x \psi_p + \psi_p \partial_t \psi_p \\ &\quad + V_p(\phi_p) + W_p(\phi_p, \psi_p, \bar{\psi}_p), \quad p = 1, 2 \\ \mathcal{L}_D &= \frac{1}{2} (\phi_2 \partial_t \phi_1 - \phi_1 \partial_t \phi_2) - \psi_1 \psi_2 - \bar{\psi}_1 \bar{\psi}_2 + 2f_1 \partial_t f_1 \\ &\quad B_0(\phi_1, \phi_2) + B_1(\phi_1, \phi_2, \psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2, f_1), \end{aligned} \quad (2.42)$$

where,

$$V_p = 4m^2 \cosh(2\phi_p), \quad W_p = 8m \bar{\psi}_p \psi_p \cosh \phi_p \quad p = 1, 2 \quad (2.43)$$

and

$$B_0 = -m\beta^2 \cosh(\phi_1 - \phi_2) - \frac{4m}{\beta^2} \cosh(\phi_1 + \phi_2), \quad (2.44)$$

$$\begin{aligned} B_1 &= -\frac{4i}{\beta} \sqrt{m} \cosh\left(\frac{\phi_1 + \phi_2}{2}\right) f_1(\bar{\psi}_1 + \bar{\psi}_2) \\ &\quad + 2i\beta \sqrt{m} \cosh\left(\frac{\phi_1 - \phi_2}{2}\right) f_1(\psi_1 - \psi_2). \end{aligned} \quad (2.45)$$

The field equations are

$x < 0$ :

$$\begin{aligned} \partial_x^2 \phi_1 - \partial_t^2 \phi_1 &= 8m \sinh(2\phi_1) + 8m \bar{\psi}_1 \psi_1 \sinh \phi_1, \\ (\partial_x - \partial_t) \bar{\psi}_1 &= 4m \psi_1 \cosh \phi_1, \\ (\partial_x + \partial_t) \psi_1 &= 4m \bar{\psi}_1 \cosh \phi_1, \end{aligned} \quad (2.46)$$

$x > 0$ :

$$\begin{aligned} \partial_x^2 \phi_2 - \partial_t^2 \phi_2 &= 8m \sinh(2\phi_2) + 8m \bar{\psi}_2 \psi_2 \sinh \phi_2, \\ (\partial_x - \partial_t) \bar{\psi}_2 &= 4m \psi_2 \cosh \phi_2, \\ (\partial_x + \partial_t) \psi_2 &= 4m \bar{\psi}_2 \cosh \phi_2, \end{aligned} \quad (2.47)$$

$x = 0$ :

$$\begin{aligned} \partial_x \phi_1 - \partial_t \phi_2 &= m\beta^2 \sinh(\phi_1 - \phi_2) + \frac{4m}{\beta^2} \sinh(\phi_1 + \phi_2) \\ &\quad - \frac{2i}{\beta} \sqrt{m} \sinh\left(\frac{\phi_1 + \phi_2}{2}\right) f_1(\bar{\psi}_1 + \bar{\psi}_2) \\ &\quad - i\beta \sqrt{m} \sinh\left(\frac{\phi_1 - \phi_2}{2}\right) f_1(\psi_1 - \psi_2). \end{aligned} \quad (2.48)$$

$$\begin{aligned} \partial_x \phi_2 - \partial_t \phi_1 &= m\beta^2 \sinh(\phi_1 - \phi_2) - \frac{4m}{\beta^2} \sinh(\phi_1 + \phi_2) \\ &\quad - \frac{2i}{\beta} \sqrt{m} \sinh\left(\frac{\phi_1 + \phi_2}{2}\right) f_1(\bar{\psi}_1 + \bar{\psi}_2) \\ &\quad - i\beta \sqrt{m} \sinh\left(\frac{\phi_1 - \phi_2}{2}\right) f_1(\psi_1 - \psi_2). \end{aligned} \quad (2.49)$$

$$\psi_1 + \psi_2 = -2i\beta \sqrt{m} \cosh\left(\frac{\phi_1 - \phi_2}{2}\right) f_1, \quad (2.50)$$

$$\bar{\psi}_1 - \bar{\psi}_2 = -\frac{4i}{\beta} \sqrt{m} \cosh\left(\frac{\phi_1 + \phi_2}{2}\right) f_1, \quad (2.51)$$

$$\begin{aligned} \partial_t f_1 &= \frac{i}{\beta} \sqrt{m} \cosh\left(\frac{\phi_1 + \phi_2}{2}\right) (\bar{\psi}_1 + \bar{\psi}_2) \\ &\quad - \frac{i\beta}{2} \sqrt{m} \cosh\left(\frac{\phi_1 - \phi_2}{2}\right) (\psi_1 - \psi_2). \end{aligned} \quad (2.52)$$

Due to (2.44) and (2.45), the first order equations (2.48)- (2.52) agrees with the Backlund transformations of the Appendix. Applying the operator  $(\partial_x + \partial_t)$  to (2.50) and using equations (2.46) and (2.47), we obtain

$$\begin{aligned}\partial_x f_1 &= \frac{i}{\beta} \sqrt{m} \cosh\left(\frac{\phi_1 + \phi_2}{2}\right) (\bar{\psi}_1 + \bar{\psi}_2) \\ &\quad + \frac{i\beta}{2} \sqrt{m} \cosh\left(\frac{\phi_1 - \phi_2}{2}\right) (\psi_1 - \psi_2).\end{aligned}\quad (2.53)$$

Equations (2.44) and (2.45) satisfy the conditions (2.12), (2.13), (2.18), (2.19) and (2.23). The modified conserved momentum and energy are given by:

$$\begin{aligned}\mathcal{P} &= P + \left[ -m\beta^2 \cosh(\phi_1 - \phi_2) + \frac{4m}{\beta^2} \cosh(\phi_1 + \phi_2) \right. \\ &\quad - \frac{4i}{\beta} \sqrt{m} \cosh\left(\frac{\phi_1 + \phi_2}{2}\right) f_1(\bar{\psi}_1 + \bar{\psi}_2) \\ &\quad \left. - 2i\beta \sqrt{m} \cosh\left(\frac{\phi_1 - \phi_2}{2}\right) f_1(\psi_1 - \psi_2) - \bar{\psi}_1 \bar{\psi}_2 + \psi_1 \psi_2 \right]_{x=0}\end{aligned}\quad (2.54)$$

$$\begin{aligned}\mathcal{E} &= E + \left[ -m\beta^2 \cosh(\phi_1 - \phi_2) - \frac{4m}{\beta^2} \cosh(\phi_1 + \phi_2) \right. \\ &\quad - \frac{4i}{\beta} \sqrt{m} \cosh\left(\frac{\phi_1 + \phi_2}{2}\right) f_1(\bar{\psi}_1 + \bar{\psi}_2) \\ &\quad \left. + 2i\beta \sqrt{m} \cosh\left(\frac{\phi_1 - \phi_2}{2}\right) f_1(\psi_1 - \psi_2) - \bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2 \right]_{x=0}\end{aligned}\quad (2.55)$$

For simplicity, we shall take  $m = 1$  from now on.

### 3 Backlund Solutions for the super Sinh-Gordon

In this section we discuss the various solutions compatible with the Backlund transformation (2.48) - (2.52) at  $x = 0$ .

- **vacuum- 1-boson system** Let  $\phi_1 = 0$ ,  $\bar{\psi}_1 = \bar{\psi}_2 = 0$  and  $\phi_2 \neq 0$ . From (2.48) and (2.49) we find

$$\phi_2 = \ln\left(\frac{1 + \frac{1}{2}b_1\rho(\sigma)}{1 - \frac{1}{2}b_1\rho(\sigma)}\right), \quad \rho(\sigma) = \exp(2\sigma(x+t) + \frac{2}{\sigma}(x-t)) \quad (3.56)$$

where  $\sigma = -\frac{2}{\beta^2}$  and  $b_1$  is an arbitrary constant.

- **vacuum- 1-fermion system** We consider solutions where  $\phi_1 = \phi_2 = 0$  and  $\psi_1 = \bar{\psi}_1 = 0$  and  $\bar{\psi}_2 \neq 0$ . From (2.50) and (2.51) we find

$$\psi_2 = -2i\beta f_1, \quad \bar{\psi}_2 = \frac{4i}{\beta} f_1. \quad (3.57)$$



Henceforth  $\psi_2 = \frac{1}{\sigma}\bar{\psi}_2$ . Eliminating  $f_1$  and  $\psi_2$  in (2.52) and (2.53) we arrive at

$$\partial_x \bar{\psi}_2 = 2(\sigma + \frac{1}{\sigma})\bar{\psi}_2, \quad \partial_t \bar{\psi}_2 = 2(\sigma - \frac{1}{\sigma})\bar{\psi}_2 \quad (3.58)$$

The solution is then given by  $\bar{\psi}_2 = \frac{2c_1\sigma}{\rho(\sigma)}$ . where  $\rho(\sigma)$  is given in (3.56) and  $c_1$  is an arbitrary grassmanian constant.

- **vacuum- fermion/boson system** We consider solutions were  $\phi_1 = \psi_1 = \bar{\psi}_1 = 0$ . From (2.48) and (2.49) we find

$$\begin{aligned} \partial_t \phi_2 &= -\frac{2i}{\beta} f_1 \sinh(\frac{\phi_2}{2}) \bar{\psi}_2 + i\beta f_1 \sinh(\frac{\phi_2}{2}) \psi_2 - 2(\frac{2}{\beta^2} - \frac{\beta^2}{2}) \sinh(\phi_2), \\ \partial_x \phi_2 &= -\frac{2i}{\beta} f_1 \sinh(\frac{\phi_2}{2}) \bar{\psi}_2 - i\beta f_1 \sinh(\frac{\phi_2}{2}) \psi_2 - 2(\frac{2}{\beta^2} + \frac{\beta^2}{2}) \sinh(\phi_2) \end{aligned} \quad (3.59)$$

Relations (2.50) and (2.51) yields

$$\psi_2 = -2i\beta \cosh(\frac{\phi_2}{2}) f_1, \quad \bar{\psi}_2 = \frac{4i}{\beta} \cosh(\frac{\phi_2}{2}) f_1 \quad (3.60)$$

and henceforth  $\psi_2 = \frac{1}{\sigma}\bar{\psi}_2$ . Eliminating  $f_1$  from (3.60) into (3.59) we find

$$\partial_t \phi_2 = 2(\sigma - \frac{1}{\sigma}) \sinh(\phi_2), \quad \partial_x \phi_2 = 2(\sigma + \frac{1}{\sigma}) \sinh(\phi_2) \quad (3.61)$$

with solution given by

$$\phi_2 = \ln \left( \frac{1 + \frac{1}{2}b_1\rho(\sigma)}{1 - \frac{1}{2}b_1\rho(\sigma)} \right). \quad (3.62)$$

Substituting now  $f_1$  in terms of  $\bar{\psi}_2, \psi_2$  and  $\phi_2$  from (3.62), we find as solution of (2.52) and (2.53)

$$\bar{\psi}_2 = -2c_1\gamma\rho \left( \frac{1}{1 - \frac{1}{4}b_1^2\rho(\sigma)^2} \right) \quad (3.63)$$

and similar for  $\psi_2$ . Similar results are obtained interchanging  $\phi_1, \psi_1, \bar{\psi}_1 \rightarrow \phi_2, \psi_2, \bar{\psi}_2$ .

Let us now change variables  $\phi_p \rightarrow \frac{i}{2}\phi_p$ ,  $x \rightarrow x/4$ ,  $t \rightarrow t/4$  in order to discuss the solutions of super sine-Gordon model.

- **boson - boson system** Let us now consider solutions with  $\bar{\psi}_1 = \bar{\psi}_2 = 0$ ,  $\phi_1 \neq 0$ ,  $\phi_2 \neq 0$ . This case yields precisely the solution obtained in [4],[5].

$$e^{i\phi_a/2} = \frac{1 - iE_a}{1 + iE_a}, \quad E_a = R_a e^{\alpha_a x + \beta_a t}, \quad (3.64)$$

where  $\alpha_a^2 - \beta_a^2 = 1$ ,  $a = 1, 2$ . The Backlund transformation (2.48) and (2.49)

$$\begin{aligned}\partial_x \phi_1 - \partial_t \phi_2 &= -\sigma \sin\left(\frac{\phi_1 + \phi_2}{2}\right) - \frac{1}{\sigma} \sin\left(\frac{\phi_1 - \phi_2}{2}\right), \\ \partial_x \phi_2 - \partial_t \phi_1 &= \sigma \sin\left(\frac{\phi_1 + \phi_2}{2}\right) - \frac{1}{\sigma} \sin\left(\frac{\phi_1 - \phi_2}{2}\right)\end{aligned}\quad (3.65)$$

imply the following relation for  $\alpha_a = \cosh \theta_a$  and  $\beta_a = \sinh \theta_a$

$$\theta_1 = \theta_2 = \theta, \quad R_2 = \left(\frac{e^\theta + \sigma}{e^\theta - \sigma}\right) R_1. \quad (3.66)$$

The defect preserves the soliton velocity, allowing at most a phase shift. However, when  $\sigma < 0$  and for  $e^\theta = |\sigma|$  we find  $\phi_2 = 0$ . This configuration corresponds to an absorption of the soliton. On the other hand, if  $\sigma > 0$  and  $e^\theta = |\sigma|$ ,  $\phi_1 = 0$  corresponding to an emission of the soliton by the defect.

- **Fermion - Fermion system** We consider solutions where  $\phi_1 = \phi_2 = 0$ . The solution of eqns. (2.46) and (2.47) are of the form

$$\bar{\psi}_a = \epsilon S_a \exp(\alpha_a x + \beta_a t), \quad \psi_a = e^{-\theta_a} \bar{\psi}_a, \quad a = 1, 2 \quad (3.67)$$

where  $\epsilon$  is a grassmanian parameter and  $\alpha_a^2 - \beta_a^2 = 1$ . From equation (2.50)-(2.53) we find

$$\begin{aligned}\frac{2}{\beta} \left( S_2 \beta_1 e^{-\theta_1} e^{\alpha_1 x + \beta_1 t} + S_2 \beta_2 e^{-\theta_2} e^{\alpha_2 x + \beta_2 t} \right) &= \frac{1}{\beta} \left( S_1 e^{\alpha_1 x + \beta_1 t} + S_2 e^{\alpha_2 x + \beta_2 t} \right) \\ &- \frac{\beta}{2} \left( S_1 e^{-\theta_1} e^{\alpha_1 x + \beta_1 t} - S_2 e^{-\theta_2} e^{\alpha_2 x + \beta_2 t} \right)\end{aligned}\quad (3.68)$$

For  $\alpha_1 \neq \alpha_2$ , it follows that  $e^{-\theta_1} = -e^{-\theta_2} = \frac{\beta^2}{2}$ . For  $\alpha_1 = \alpha_2$ , we find  $\theta_1 = \theta_2 = \theta$  and

$$S_2 = \left(\frac{e^\theta + \sigma}{e^\theta - \sigma}\right) S_1 \quad (3.69)$$

where  $\alpha_a = \cosh \theta$ ,  $\beta_a = \sinh \theta$ . Again the velocity is preserved with only a phase shift being allowed when the soliton interacts with the defect. Notice that limiting cases, where  $S_2 = 0$  or  $S_1 = 0$  are obtained when  $\sigma < 0$  and  $e^\theta = |\sigma|$  or  $\sigma > 0$  and  $e^\theta = \sigma$ . These cases correspond to total absorption or creation of one fermion respectively.

- **Fermion/Boson - Fermion/Boson system**

Consider the following solution of eqns. (2.46) and (2.47)

$$\begin{aligned}e^{i\phi_a} &= \frac{1 - iE_a}{1 + iE_a}, \quad E_a = R_a e^{\alpha_a x + \beta_a t}, \quad \alpha_a^2 - \beta_a^2 = 1, a = 1, 2 \\ \bar{\psi}_a &= \epsilon S_a e^{\alpha_a x + \beta_a t} \left( \frac{1}{1 + iE_a} + \frac{1}{1 - iE_a} \right), \quad \psi_a = e^{-\theta_a} \bar{\psi}_a\end{aligned}\quad (3.70)$$

We now substitute (3.67) and (3.69) into equations (2.48) - (2.53),

$$\begin{aligned}
\partial_x \phi_1 - \partial_t \phi_2 &= -\sigma \sin\left(\frac{\phi_1 + \phi_2}{2}\right) - \frac{1}{\sigma} \sin\left(\frac{\phi_1 - \phi_2}{2}\right), \\
\partial_x \phi_2 - \partial_t \phi_1 &= \sigma \sin\left(\frac{\phi_1 + \phi_2}{2}\right) - \frac{1}{\sigma} \sin\left(\frac{\phi_1 - \phi_2}{2}\right), \\
(\psi_1 + \psi_2) \cos\left(\frac{\phi_1 + \phi_2}{4}\right) &= -\frac{1}{\sigma} (\bar{\psi}_1 - \bar{\psi}_2) \cos\left(\frac{\phi_1 - \phi_2}{4}\right).
\end{aligned} \tag{3.71}$$

Writing  $\alpha_a = \cosh \theta_a$ ,  $\beta_a = \sinh \theta_a$  in (3.70) we find from (3.71)

$$\theta_1 = \theta_2 = \theta, \quad S_2 = \left( \frac{e^\theta + \sigma}{e^\theta - \sigma} \right) S_1, \quad R_2 = \left( \frac{e^\theta + \sigma}{e^\theta - \sigma} \right) R_1 \tag{3.72}$$

Again the velocity is preserved with only a phase shift being allowed when the soliton interacts with the defect. Notice that limiting cases, where  $S_2 = R_2 = 0$  or  $S_1 = R_1 = 0$  are obtained when  $\sigma < 0$  and  $e^\theta = |\sigma|$  or  $\sigma > 0$  and  $e^\theta = \sigma$ . These cases correspond to total absorption or creation of one soliton respectively.

## 4 Zero Curvature Formulation

Consider the  $sl(2, 1)$  super Lie algebra with generators

$$h_1 = \alpha_1 \cdot H, \quad h_2 = \alpha_2 \cdot H, \quad E_{\pm\alpha_1}, \quad E_{\pm\alpha_2}, \quad E_{\pm(\alpha_1+\alpha_2)} \tag{4.73}$$

where  $\alpha_1$  and  $\alpha_2, \alpha_1 + \alpha_2$  are bosonic and fermionic roots respectively. We now extend the finite dimensional Lie algebra  $sl(2, 1)$  to an affine structure by introducing the spectral parameter  $\lambda$  as follows,

$$T_a \rightarrow T_a^{(n)} = \lambda^n T_a, \quad \lambda \in C \tag{4.74}$$

where  $n \in Z$  or  $n \in Z + 1/2$  according to  $T_a$  denoting bosonic or fermionic generators respectively [9].

The super sinh-Gordon model can be described by the Lax pair (see for instance ref. [9])

$$\begin{aligned}
a_t^{(p)} &= -\frac{1}{2} \partial_x \phi_p h_1 + \left( \lambda - \frac{1}{\lambda} \right) (h_1 + 2h_2) + \left( \frac{e^{\phi_p}}{\lambda} - \lambda e^{-\phi_p} \right) E_{\alpha_1} + \left( \frac{e^{-\phi_p}}{\lambda} - \lambda e^{\phi_p} \right) E_{-\alpha_1} \\
&+ \left( \lambda^{\frac{1}{2}} e^{\frac{1}{2}\phi_p} \bar{\psi}_p + \lambda^{-\frac{1}{2}} e^{-\frac{1}{2}\phi_p} \psi_p \right) E_{\alpha_2} + \left( \lambda^{\frac{1}{2}} e^{-\frac{1}{2}\phi_p} \bar{\psi}_p - \lambda^{-\frac{1}{2}} e^{\frac{1}{2}\phi_p} \psi_p \right) E_{-\alpha_2} \\
&+ \left( \lambda^{\frac{1}{2}} e^{-\frac{1}{2}\phi_p} \bar{\psi}_p + \lambda^{-\frac{1}{2}} e^{\frac{1}{2}\phi_p} \psi_p \right) E_{\alpha_1+\alpha_2} + \left( \lambda^{\frac{1}{2}} e^{\frac{1}{2}\phi_p} \bar{\psi}_p - \lambda^{-\frac{1}{2}} e^{-\frac{1}{2}\phi_p} \psi_p \right) E_{-\alpha_1-\alpha_2} \\
\\
a_x^{(p)} &= -\frac{1}{2} \partial_t \phi_p h_1 + \left( \lambda + \frac{1}{\lambda} \right) (h_1 + 2h_2) - \left( \frac{e^{\phi_p}}{\lambda} + \lambda e^{-\phi_p} \right) E_{\alpha_1} - \left( \frac{e^{-\phi_p}}{\lambda} + \lambda e^{\phi_p} \right) E_{-\alpha_1} \\
&+ \left( \lambda^{\frac{1}{2}} e^{\frac{1}{2}\phi_p} \bar{\psi}_p - \lambda^{-\frac{1}{2}} e^{-\frac{1}{2}\phi_p} \psi_p \right) E_{\alpha_2} + \left( \lambda^{\frac{1}{2}} e^{-\frac{1}{2}\phi_p} \bar{\psi}_p + \lambda^{-\frac{1}{2}} e^{\frac{1}{2}\phi_p} \psi_p \right) E_{-\alpha_2} \\
&+ \left( \lambda^{\frac{1}{2}} e^{-\frac{1}{2}\phi_p} \bar{\psi}_p - \lambda^{-\frac{1}{2}} e^{\frac{1}{2}\phi_p} \psi_p \right) E_{\alpha_1+\alpha_2} + \left( \lambda^{\frac{1}{2}} e^{\frac{1}{2}\phi_p} \bar{\psi}_p + \lambda^{-\frac{1}{2}} e^{-\frac{1}{2}\phi_p} \psi_p \right) E_{-\alpha_1-\alpha_2}
\end{aligned} \tag{4.75}$$

$p = 1, 2$

where the power of  $\lambda$  denotes the effective grading and henceforth the generators  $h_i, E_{\pm\alpha}$  correspond to  $h_i^{(0)}, E_{\pm\alpha}^{(0)}$  respectively. The corresponding zero curvature equation yields the equations of motion (2.46) and (2.47) with  $m = 1$ .

In order to describe the integrability of the system, we follow ref. [6] and split the space into two overlapping regions, namely,  $x \leq b$  and  $x \geq a$  with  $a < b$ . Inside the overlap region, i.e.,  $a \leq x \leq b$ , define the Lax pair to be,

$$\begin{aligned}
\hat{a}_t^{(1)} &= a_t^{(1)} - \frac{1}{2}\theta(x-a)((\partial_x\phi_1 - \partial_t\phi_2 + \partial_{\phi_1}B_0 + \partial_{\phi_1}B_1)h_1 \\
&\quad + (\psi_1 + \psi_2 - \partial_{\psi_1}B_1)E_1 + (\partial_tf_1 + \frac{1}{4}\partial_{f_1}B_1)E'_1), \\
\hat{a}_x^{(1)} &= \theta(a-x)a_x^{(1)}, \\
\hat{a}_t^{(2)} &= a_t^{(2)} - \frac{1}{2}\theta(b-x)((\partial_x\phi_2 - \partial_t\phi_1 - \partial_{\phi_2}B_0 - \partial_{\phi_2}B_1)h_1 \\
&\quad + (\bar{\psi}_1 - \bar{\psi}_2 + \partial_{\bar{\psi}_2}B_1)E_2 + (\partial_tf_1 + \frac{1}{4}\partial_{f_1}B_1)E'_2), \\
\hat{a}_x^{(2)} &= \theta(x-b)a_x^{(2)}.
\end{aligned} \tag{4.76}$$

where  $E_i$  and  $E'_i$  denote a pair of independent fermionic step operators of  $sl(2,1)$ . Within the overlap region, the Lax pairs denoted by the suffices  $p = 1, 2$  are related by gauge transformation. In particular for the time component  $a_t^{(p)}$ ,

$$K^{-1}\partial_t K = \hat{a}_t^{(2)} - K^{-1}\hat{a}_t^{(1)}K \tag{4.77}$$

If we now decompose  $K$  into

$$K = e^{\frac{1}{2}\phi_2 h_1} \bar{K} e^{-\frac{1}{2}\phi_1 h_1}, \quad \bar{K} = e^{-m(-1/2) - m(-1) - \dots} \tag{4.78}$$

we find

$$\begin{aligned}
m(-1/2) &= \frac{2if_1}{\beta}\lambda^{-\frac{1}{2}}(E_{\alpha_2} - E_{-\alpha_2} + E_{\alpha_1+\alpha_2} - E_{-\alpha_1-\alpha_2}) \\
m(-1) &= c\lambda^{-1}(h_1 + 2h_2) - \frac{2\lambda^{-1}}{\beta^2}(E_{\alpha_1} + E_{-\alpha_1})
\end{aligned} \tag{4.79}$$

provided the Backlund transformation (2.46)- (2.53) holds and  $c$  is an arbitrary constant which, for convenience is chosen as  $c = \frac{2}{\beta^2}$ . It then follows that  $m(-3/2) = m(-2) = 0$ ,

$$m(-5/2) = \frac{1}{3}\left(\frac{2}{\beta^2}\right)^2\lambda^{-2}m(-1/2), \quad m(-3) = \frac{1}{3}\left(\frac{2}{\beta^2}\right)^2\lambda^{-2}m(-1), \dots \tag{4.80}$$

A general closed solution can be put in the following form,

$$\bar{K} = I(1 + \frac{2}{\lambda\beta^2})\Lambda - m(-1/2)\Lambda - m(-1)\Lambda \tag{4.81}$$

where  $\Lambda$  is an arbitrary constant. By choosing  $\Lambda = (1 + \frac{2}{\lambda\beta^2})^{-1}$  we recover (4.79) and (4.80) after expansion in powers of  $\frac{1}{\lambda}$  of  $\Lambda$ ,

$$\bar{K} = I - m(-1/2) + \frac{2}{\lambda\beta^2}m(-1/2) + \dots - m(-1) + \frac{2}{\lambda\beta^2}m(-1) + \dots \tag{4.82}$$

which can be rewritten as in (4.78), i.e.,

$$\bar{K} = \exp \left( -m(-1/2) - m(-1) - \frac{1}{3} \left( \frac{2}{\beta^2} \right)^2 \lambda^{-2} m(-1/2) - \dots \right) \quad (4.83)$$

provided the following relations hold

$$\begin{aligned} \frac{2}{\lambda\beta^2} m(-1) &= \frac{1}{2} m(-1)m(-1) \\ \frac{2}{\lambda\beta^2} m(-1/2) &= \frac{1}{2} (m(-1/2)m(-1) + m(-1)m(-1/2)), \\ \left( \frac{2}{\lambda\beta^2} \right)^2 m(-1/2) &= \frac{1}{4} (m(-1)m(-1/2)m(-1) + m(-1)m(-1)m(-1/2) \\ &\quad + m(-1/2)m(-1)m(-1)) \end{aligned} \quad (4.84)$$

These are verified using 3 dimensional matrix representation of the algebra  $sl(2, 1)$ .

The existence of the gauge transformation (4.77) provides a generating function for an infinite set of constants of motion (see [5]) strongly indicating the integrability of the system.

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## 5 Appendix - Backlund Transformation in components

Let  $\Phi$  be a bosonic super field

$$\Phi = \phi + \theta_1 \bar{\psi} + i\theta_2 \psi - \theta_1 \theta_2 F \quad (5.85)$$

and

$$D_z = \partial_{\theta_1} + \theta_1 \partial_z, \quad D_{\bar{z}} = \partial_{\theta_2} + \theta_2 \partial_{\bar{z}}, \quad D_z^2 = \partial_z, \quad D_{\bar{z}}^2 = \partial_{\bar{z}}, \quad D_z D_{\bar{z}} = -D_{\bar{z}} D_z \quad (5.86)$$

are the corresponding supersymmetric covariant derivatives. Here the space-time is defined in terms of the light cone coordinates as  $x = \frac{1}{2}(z + \bar{z})$ ,  $t = \frac{1}{2}(z - \bar{z})$  and therefore  $\partial_z = \frac{1}{2}(\partial_x + \partial_t)$ ,  $\partial_{\bar{z}} = \frac{1}{2}(\partial_x - \partial_t)$ . The equation of motion in terms of the superfield (5.85) is

$$D_z D_{\bar{z}} \Phi = 2i \sinh \Phi \quad (5.87)$$

if the auxiliary field  $F$  satisfies  $F = 2i \sinh \phi$ . According to Chaichian and Kulish [7], the Backlund tranformation for the supersymmetric sinh-Gordon is given by the following first order equations

$$\begin{aligned} D_z \Phi_1 &= D_z \Phi_2 - \frac{4i}{\beta} f \cosh \left( \frac{\Phi_1 + \Phi_2}{2} \right) \\ D_{\bar{z}} \Phi_1 &= -D_{\bar{z}} \Phi_2 + 2\beta f \cosh \left( \frac{\Phi_1 - \Phi_2}{2} \right) \end{aligned} \quad (5.88)$$

Their compatibility imply in (5.87) if the fermionic super field  $f$  satisfies

$$D_{\bar{z}}f = \beta \sinh\left(\frac{\Phi_1 - \Phi_2}{2}\right), \quad D_z f = \frac{2i}{\beta} \sinh\left(\frac{\Phi_1 + \Phi_2}{2}\right) \quad (5.89)$$

Let  $f$  be written as  $f = f_1 + \theta_1 b_1 + \theta_2 b_2 + \theta_1 \theta_2 f_2$  where  $b_1, b_2$  are bosonic fields and  $f_1, f_2$  are fermionic fields and denote

$$\phi_{\pm} = \phi_1 \pm \phi_2, \quad F_{\pm} = F_1 \pm F_2, \quad \bar{\psi}_{\pm} = \bar{\psi}_1 \pm \bar{\psi}_2, \quad \psi_{\pm} = \psi_1 \pm \psi_2. \quad (5.90)$$

In components, (5.88) and (5.89) can be written as

$$f_1 = -\frac{\beta \bar{\psi}_-}{4i \cosh\left(\frac{\phi_+}{2}\right)} = \frac{i\psi_+}{2\beta \cosh\left(\frac{\phi_-}{2}\right)}, \quad (5.91)$$

$$\partial_{\bar{z}} f_1 = \frac{i\beta}{2} \cosh\left(\frac{\phi_-}{2}\right) \psi_-, \quad (5.92)$$

$$\partial_z f_1 = \frac{i}{\beta} \cosh\left(\frac{\phi_+}{2}\right) \bar{\psi}_+, \quad (5.93)$$

$$b_1 = \frac{2i}{\beta} \sinh\left(\frac{\phi_+}{2}\right), \quad (5.94)$$

$$\partial_{\bar{z}} b_1 = -\beta \left[ -\frac{1}{2} \cosh\left(\frac{\phi_-}{2}\right) F_- + \frac{i}{4} \sinh\left(\frac{\phi_-}{2}\right) \psi_- \bar{\psi}_- \right], \quad (5.95)$$

$$b_2 = \beta \sinh\left(\frac{\phi_-}{2}\right), \quad (5.96)$$

$$\partial_z b_2 = \frac{2i}{\beta} \left[ -\frac{1}{2} \cosh\left(\frac{\phi_+}{2}\right) F_+ + \frac{i}{4} \sinh\left(\frac{\phi_+}{2}\right) \psi_+ \bar{\psi}_+ \right], \quad (5.97)$$

$$f_2 = -\frac{\beta}{2} \cosh\left(\frac{\phi_-}{2}\right) \bar{\psi}_- = -\frac{1}{\beta} \cosh\left(\frac{\phi_+}{2}\right) \psi_+, \quad (5.98)$$

$$F_- = -\frac{4i}{\beta} \left[ \frac{f_1}{2} \sinh\left(\frac{\phi_+}{2}\right) \psi_+ - b_2 \cosh\left(\frac{\phi_+}{2}\right) \right], \quad (5.99)$$

$$F_+ = -2\beta \left[ \frac{f_1}{2} \sinh\left(\frac{\phi_-}{2}\right) \bar{\psi}_- - b_1 \cosh\left(\frac{\phi_-}{2}\right) \right], \quad (5.100)$$

$$\partial_z \phi_- = \frac{4i}{\beta} \left[ \frac{f_1}{2} \sinh\left(\frac{\phi_+}{2}\right) \bar{\psi}_+ - b_1 \cosh\left(\frac{\phi_+}{2}\right) \right], \quad (5.101)$$

$$\partial_{\bar{z}} \phi_+ = -2\beta \left[ \frac{if_1}{2} \sinh\left(\frac{\phi_-}{2}\right) \psi_- - b_2 \cosh\left(\frac{\phi_-}{2}\right) \right], \quad (5.102)$$

$$\begin{aligned} i\partial_z \psi_- &= \frac{4i}{\beta} \left[ \frac{f_1}{2} \sinh\left(\frac{\phi_+}{2}\right) F_+ - \frac{if_1}{4} \cosh\left(\frac{\phi_+}{2}\right) \psi_+ \bar{\psi}_+ \right] \\ &\quad - \frac{4i}{\beta} \left[ \frac{ib_1}{2} \sinh\left(\frac{\phi_+}{2}\right) \psi_+ - \frac{b_2}{2} \sinh\left(\frac{\phi_+}{2}\right) \bar{\psi}_+ \right] \\ &\quad - \frac{4i}{\beta} f_2 \cosh\left(\frac{\phi_+}{2}\right), \end{aligned} \quad (5.103)$$

$$\begin{aligned}
\partial_{\bar{z}}\bar{\psi}_+ &= -2\beta \left[ -\frac{f_1}{2} \sinh\left(\frac{\phi_-}{2}\right) F_- + \frac{if_1}{4} \cosh\left(\frac{\phi_-}{2}\right) \psi_- \bar{\psi}_- \right] \\
&\quad -2\beta \left[ \frac{ib_1}{2} \sinh\left(\frac{\phi_-}{2}\right) \psi_- - \frac{b_2}{2} \sinh\left(\frac{\phi_-}{2}\right) \bar{\psi}_- \right] \\
&\quad -2\beta f_2 \cosh\left(\frac{\phi_-}{2}\right).
\end{aligned} \tag{5.104}$$

$$\tag{5.105}$$

Eliminating the unphysical fields, we find

$$\partial_z \phi_- = -\frac{1}{2} \tanh\left(\frac{\phi_+}{2}\right) \bar{\psi}_- \bar{\psi}_+ + \frac{4}{\beta^2} \sinh \phi_+, \tag{5.106}$$

$$\partial_z \bar{\psi}_- = \frac{1}{2} \bar{\psi}_- \tanh\left(\frac{\phi_+}{2}\right) \partial_z \phi_+ + \frac{4}{\beta^2} \cosh^2\left(\frac{\phi_+}{2}\right) \bar{\psi}_+ \tag{5.107}$$

$$\partial_{\bar{z}} \phi_- = \frac{\beta^2}{8} \frac{\tanh(\phi_-/2)}{\cosh^2(\phi_+/2)} \bar{\psi}_- \partial_t \bar{\psi}_- + \beta^2 \sinh \phi_-, \tag{5.108}$$

$$\begin{aligned}
\partial_{\bar{z}} \bar{\psi}_+ &= \tanh\left(\frac{\phi_+}{2}\right) \tanh\left(\frac{\phi_-}{2}\right) \partial_{\bar{z}} \bar{\psi}_- \\
&\quad - \frac{\beta^2}{2} \tanh^2\left(\frac{\phi_+}{2}\right) \tanh\left(\frac{\phi_-}{2}\right) \sinh \phi_- \bar{\psi}_- \\
&\quad + \beta^2 \cosh^2\left(\frac{\phi_-}{2}\right) \bar{\psi}_-.
\end{aligned} \tag{5.109}$$

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